

# A COMPARISON STUDY OF MODAL PARAMETER CONFIDENCE INTERVALS COMPUTED USING THE MONTE CARLO AND BOOTSTRAP TECHNIQUES

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**ABSTRACT.** *This paper presents a comparison of two techniques used to estimate the statistical confidence intervals on modal parameters identified from measured vibration data. The first technique is Monte Carlo simulation, which involves the repeated simulation of random data sets based on the statistics of the measured data and an assumed distribution of the variability in the measured data. A standard modal identification procedure is repeatedly applied to the randomly perturbed data sets to form a statistical distribution on the identified modal parameters. The second technique is the Bootstrap approach, where individual frequency response function (FRF) measurements are randomly selected with replacement to form an ensemble average. This procedure, in effect, randomly weights the various FRF measurements. These weighted averages of the FRFs are then put through the modal identification procedure. The modal parameters identified from each randomly weighted data set are then used to define a statistical distribution for these parameters. The basic difference in the two techniques is that the Monte Carlo technique requires the assumption on the form of the distribution of the variability in the measured data, while the bootstrap technique does not. Also, the Monte Carlo technique can only estimate random errors, while the bootstrap statistics represent both random and bias (systematic) variability such as that arising from changing environmental conditions. However, the bootstrap technique requires that every frequency response function be saved for each average during the data acquisition process. Neither method can account for bias introduced during the estimation of the FRFs. The confidence intervals resulting from the applications of the two techniques to modal properties identified from frequency response function data measured on the Alamosa Canyon bridge in southern New Mexico will be presented and compared.*

## 1. INTRODUCTION.

An important element of any experimental procedure is the ability to quantify the uncertainty in the test results that can be attributed to either random or bias (systematic) variability. However, a review of the experimental modal analysis literature shows relatively few studies aimed at developing methods to quantify the uncertainty in estimated modal parameters. References 1- 6 are some examples of these studies. To the authors' knowledge, attempts at comparing different statistical methods on similar experimental modal analysis data sets have not been published. This paper will summarize such a study that was performed on modal data obtained from a bridge structure.

This study has been motivated by a program to develop vibration-based damage identification procedures. Highway bridges have been the primary structure studied in this investigation. Several investigators [7-11] have examined the variability in identified modal parameters for bridge structures caused by environmental effects and other systematic sources of variability. These sources of variability were shown to produce significant changes in the identified modal properties of the structure. However, the prime motive for exploring this issue resulted from a study where various damage ID methods were applied to data from a bridge structure and subsequently to numerical simulations of the experiment [12]. Ambiguous results were obtained where the methods did not work as well on the noise-free numerical data generated from the finite element model. These results led the authors to search for methods to statistically quantify the uncertainties in the identified modal parameters and subsequently to quantify uncertainties in the damage ID parameters [13]. As a result, the authors believe that any vibration-based damage detection method can not be applied with confidence unless

the variability in the vibration response of the structure can be quantified in some manner.

Reference 14 reports the authors' first attempt at generating a general procedure for analyzing the uncertainty in identified modal parameters. This reference summarizes the development of a Monte Carlo simulation procedure to quantify the 95% confidence limits on the identified modal parameters. The study reported herein extends this work by implementing a bootstrap statistical analysis procedure and comparing the results obtained with it to those obtained from the Monte Carlo analysis procedure. The two statistical analysis procedures are first described followed by a summary of their application to a numerically generated set of randomly perturbed vibration data. These procedures are then applied to two sets of data measured on a bridge structure. These two data sets represent a case where random variability is assumed to be the primary source of variability as well as a case where systematic variability is known to be present.

## 2. MONTE CARLO ANALYSIS

The first step in the Monte Carlo Analysis procedure is to establish statistical uncertainty bounds on the measured frequency response function (FRF) magnitude and phase. The procedure developed by Bendat and Piersol [15] for a randomly excited single input/single output model is employed for this estimate. This procedure assumes that the variability is random and distributed in a Gaussian manner. A further assumption is made that the record lengths are long enough such that resolution bias errors are negligible. The analysis presented by Bendat and Piersol leads to the following relations for the standard deviation,  $\sigma$ , on the magnitude and phase for the FRFs at each frequency value,  $\omega$ :

$$\sigma(|H(\omega)|) = \frac{\sqrt{1 - \gamma_{mean}^2(\omega)}}{|\gamma_{mean}(\omega)|\sqrt{2n_d}} |H_{mean}(\omega)|, \text{ and} \quad (1)$$

$$\sigma(\angle H(\omega)) = \frac{\sqrt{1 - \gamma_{mean}^2(\omega)}}{|\gamma_{mean}(\omega)|\sqrt{2n_d}} \angle H_{mean}(\omega). \quad (2)$$

In Eqs. 1 and 2  $|H_{mean}(\omega)|$  is the mean value of the FRF magnitude at frequency  $\omega$ ,  $\angle H_{mean}(\omega)$  is the corresponding mean value of the FRF phase angle,  $\gamma_{mean}(\omega)$  is the mean value of the coherence function,  $n_d$  is the number of measurement averages used to form the mean values of the FRF and coherence.

Monte Carlo analyses [16] were then performed, using the previously determined uncertainty bounds on the FRFs, to establish statistical uncertainty bounds on the identified modal parameters (frequencies, damping ratios, and mode shapes). The basic idea of a Monte Carlo analysis is the

repeated simulation of random input data, in this case the FRFs with estimated mean and standard deviation values, and compilation of statistics on the output data, in this case the rational polynomial curve-fit [17] results. The steps followed to implement this procedure are:

1. Measure the averaged FRF and coherence functions for each degree of freedom (DOF).
2. At each frequency interval random Gaussian noise is added to the magnitude and phase of the mean FRF for all measurements. This noise is based on the mean values of the FRFs obtained from the ensemble averages made during the measurement process and standard deviation values calculated from Eqs. 1 and 2. A unique random number is used to perturb each FRF at each discrete frequency.
3. The rational polynomial curve fit is applied to the perturbed FRFs and standard modal properties (resonant frequencies, modal damping, mode shape amplitudes and phases) are identified.
4. Steps 1, 2, and 3 are repeated and the mean and standard deviation for the identified modal properties are calculated at the end of each iteration. The process is repeated until the means and standard deviations converge (typically less than 100 iterations).

Ninety-five percent confidence limits for each modal parameter are established based on the mean and standard deviation values calculated from the Monte Carlo simulations along with the assumption of the Gaussian distribution on the variations in these parameters. Note that this procedure does not require storage of individual FRF measurements that make up the ensemble averages at each DOF.

## 3. BOOTSTRAP ANALYSIS

The bootstrap analysis procedure [18] randomly selects individual FRF measurements at each DOF to form the ensemble average. Because the FRFs for a particular DOF are selected at random and "with replacement," a single FRF sample may be used more than once in the ensemble average while others may not be included. This process results in ensemble averages that are based on random weighting of the sample FRFs.

In application, the same sequence of randomly selected sample FRFs was used for each DOF. This selection procedure preserves the temporal variations in measured inputs and responses across all DOFs. As an example, assume that a large truck on an adjacent bridge applies a significant unmeasured input into the bridge that is being testing during a particular FRF measurement. The selection process described above will assure that all DOFs include this measurement with equal weighting. Such a weighting scheme is important when attempting to accurately assess the variations of modal parameters using global curve-fitting algorithms (algorithms that estimate modal parameter by

fitting all FRFs simultaneously). This selection process should accurately account for variations that are a function of spatial location such as a particularly noisy sensor.

Once the randomly weighted ensemble averages are formed for each DOF, the rational polynomial parameter estimation procedure is applied and the modal parameters are identified. This procedure is repeated numerous times to form a histogram of the identified modal parameters.

The percentile interval method discussed in [18] is then used to establish the 95 percent confidence limits on the identified modal parameters. The values of the parameters are placed in an ordered list. Values of the parameters corresponding to the upper and lower 2.5 percentile are used to define the 95 percent confidence limits.

The steps used to implement the bootstrap procedure are:

1. Measure and store N FRF samples at each DOF.
2. At each DOF randomly select N FRFs, with replacement, and form an ensemble average. The same random sequence of FRFs is used at each DOF.
3. Apply the rational polynomial curve-fit procedure to these ensemble averages to determine the modal parameters (resonant frequencies, modal damping, mode shape amplitudes and phases).
4. Repeat steps 1-3 and form a histogram for each modal parameter.
5. Calculate the mean value of each modal parameter. Place the parameters into an ordered list and estimate the 95 percent confidence limits from the values corresponding to the upper and lower 2.5 percentiles.

Note that no assumption on the distribution of the identified parameters, no assumption about the form of the input or response, and no assumption about the number of inputs or responses are made in this analysis. The bootstrap method does require each individual FRF to be stored for each measurement DOF. As with the Monte Carlo method, no quantitative method was employed to determine the number of iterations needed for the bootstrap method to converge. Rather, various numbers of iterations were tried until there was no significant change in the confidence limits calculated for the various parameters. Typically, approximately 100 iterations were required for convergence.

Finally, it should be noted that both the Monte Carlo and the bootstrap procedures will not account for bias errors introduced by procedures to estimate the FRFs (for example, those introduced by windowing functions) or for bias errors introduced during the parameter estimation procedure.

#### 4. TEST ON SIMULATED DATA

To test the two statistical analysis procedures, a set of 30 randomly perturbed FRFs were generated for a single degree of freedom system. This idealized system has a

mean resonant frequency and damping ratio of 2.5 Hz and 1%, respectively. A standard deviation of 0.002 Hz was assigned to the resonant frequency and a standard deviation of 0.08% was assigned to the damping value. These frequency and damping values along with the corresponding standard deviations are based on results obtained from bridge structures [11]. A unit mass was assigned to this system. The generated FRFs,  $H(i\omega)$ , were defined at 0.03125 Hz increments and have the form

$$H(i\omega) = \frac{C}{i\omega - \lambda} + \frac{C^*}{i\omega - \lambda^*}, \quad (3)$$

where  $i = \sqrt{-1}$ ,

$\omega$  = cyclic frequency,

$$C = \frac{1}{2iM\omega_d} (\text{residue}),$$

$M$  = the system modal mass,

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \text{ (damped natural frequency),}$$

$\omega_n$  = system natural frequency,

$\zeta$  = system damping ratio,

$\lambda = \zeta\omega_n + \omega_d i$  (pole), and

\* denotes complex conjugate.

Note that standard deviation on the resonant frequency is smaller than one frequency interval. An alternate case was studied where the standard deviation of the resonant frequency was 0.1 Hz. However, the example reported is considered more severe. Care must be taken in generating variations in damping such that negative damping values are not specified. Prior to performing the statistical analyses, the rational polynomial curve-fitting algorithm was applied to each of the 30 generated FRFs. The curve-fitting algorithm predicted the specified parameters almost exactly, hence it can be concluded that for this simulated data no bias error is being introduced by this algorithm. Further studies are needed to determine if this statement will still hold for multi degree-of-freedom systems with closely spaced modes.

Figure 1 shows an overlay of the 30 FRF magnitudes generated for this system. The variations in the FRFs can only be seen when the plot is expanded around the resonant frequency. Coherence functions were not generated for the implementation of the Monte Carlo procedure. Instead, the 30 FRFs were used to generate a mean and standard deviation for the real and imaginary part at each frequency. The Monte Carlo procedure then generated perturbed FRFs based on these statistics and the assumption of a Gaussian distribution.

Table 1 summarizes the results of applying the Monte Carlo and bootstrap statistical analysis methods to the numerically generated data. **This example shows that for a case**

where the variations in the measured data correspond to the assumptions that the Monte Carlo method is based upon, the procedure accurately predicts the uncertainty in the parameters identified from the simulated data. Similarly, the bootstrap analysis also predicts the correct uncertainty in the

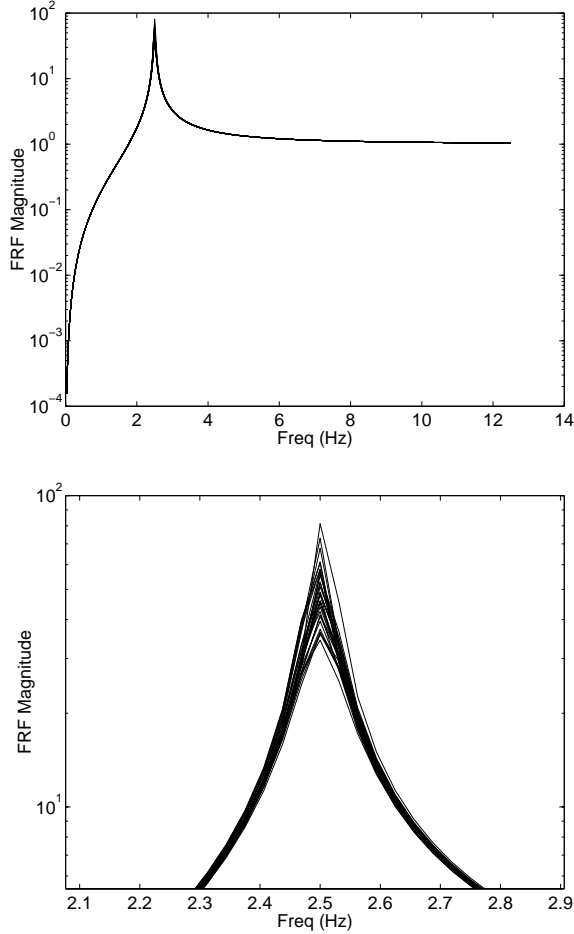


Fig. 1 Simulated FRFs.

Method (Iteration)	$\omega_{\text{mean}}$	$\omega$ 95% Conf. Intervals	$\zeta_{\text{mean}}$	$\zeta$ 95% Conf. Intervals
True Value	2.50 Hz	0.0037 Hz	1.00%	0.18%
BS (100)	2.50 Hz	0.0040 Hz	1.00%	0.16%
BS (500)	2.50 Hz	0.0041 Hz	1.00%	0.17%
BS (1000)	2.50 Hz	0.0039 Hz	1.00%	0.17%
MC (50)	2.50 Hz	0.0032 Hz	1.00%	0.16%
MC (75)	2.50 Hz	0.0036 Hz	0.99%	0.15%
MC (100)	2.50 Hz	0.0036 Hz	0.98%	0.16%

BS = Bootstrap, MC = Monte Carlo

modal parameters. Note that although a 95% confidence interval of 0.004 Hz was specified the actual

data used to generate the 30 FRFs had a 95% confidence interval of 0.0037 Hz. Similarly, the actual 95% confidence limits on the damping is 0.18%.

Once there was confidence that the statistical analysis procedures were working correctly, these methods were applied to data measured on a bridge structure.

## 5. ALAMOSA CANYON BRIDGE

The Alamosa Canyon Bridge has seven independent spans with a common pier between successive spans. An elevation

View of the bridge is shown in Fig.2. Each span consists of a concrete deck supported by six W30x116 steel girders. The roadway in each span is approximately 7.3 m (24 ft) wide and 15.2 (50 ft) long. A concrete curb and guardrail are integrally attached to the deck. Four sets of cross braces are equally spaced along the length of the span between adjacent girders. The cross braces are channel sections (C12x25). A cross section of the span at the interior cross braces is shown in Fig. 3. At the pier the girders rest on rollers as shown in Fig. 4. Also shown in Fig. 4 is the connection detail at the abutment where the beams are bolted to a half-roller to simulate a pinned connection. The bridge is aligned in a north-south direction.

## 6. DATA ACQUISITION

The data acquisition system and measurement hardware described in [11] were set up to measure acceleration- and force-time histories. FRFs and coherence functions were then calculated from the measured quantities. Sampling parameters were specified that calculated the FRFs from a 16-s time window discretized with 2048 samples. The FRFs were calculated for a frequency range of 0 to 50 Hz. These sampling parameters produced a frequency resolution of 0.0625 Hz. A Force window was applied to the signal from the hammer's force transducer and exponential windows were applied to the signals from the accelerometer. AC coupling was specified to minimize DC offsets.

A total of 31 acceleration measurements were made on the concrete deck and on the girders below the bridge as shown in Fig. 5. Five accelerometers were spaced along the length of each girder. Because of the limited number of data channels measurements were not made on the girders at the abutment or at the pier. Excitation was applied in the vertical direction on the top surface of the deck with an instrumented hammer. The force-input and acceleration-response time histories obtained from each impact were subsequently transformed into the frequency domain so that estimates of the FRFs and coherence functions could be calculated. Thirty averages were used for these estimates. With the sampling parameters listed above and the overload reject specified, data acquisition for a specific test usually occurred over a time period of approximately 30 - 45 minutes.

## 7. APPLICATION TO THE ALAMOSA CANYON BRIDGE

Two sets of FRFs were investigated for the Alamosa Canyon Bridge. The first set of data consisted of 30 averages made early in the morning before the sun hit the bridge. These measurements were made over a 30-minute time interval. Air temperatures remained very constant during the time of the test. Traffic on the adjacent bridges was light during this period. Therefore, it was assumed that

variability in the FRF measurements made during this test were the result of random error sources. Figure 6 shows an overlay of the thirty FRF magnitudes measured at location 2 during this test. Figure 7 shows the average FRF for location 2. Figure 8 shows the corresponding coherence function that was used to generate the standard deviations associated with these measurements.



Figure 2. Elevation View of the Alamosa Canyon Bridge

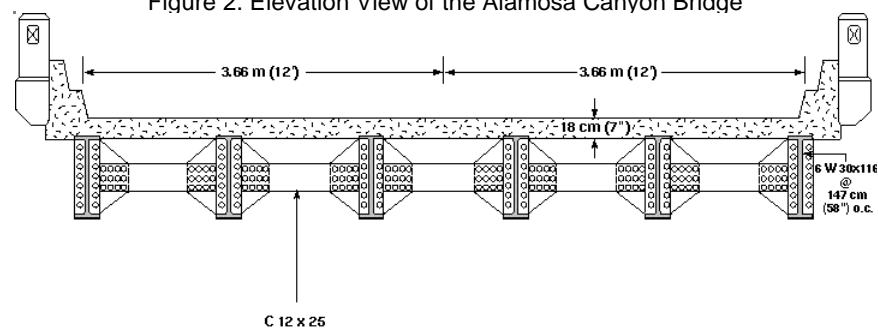


Figure 3. Cross-section view of the Alamosa Canyon Bridge.



Pier



Abutment

Figure 4. Support details at the pier and abutment.

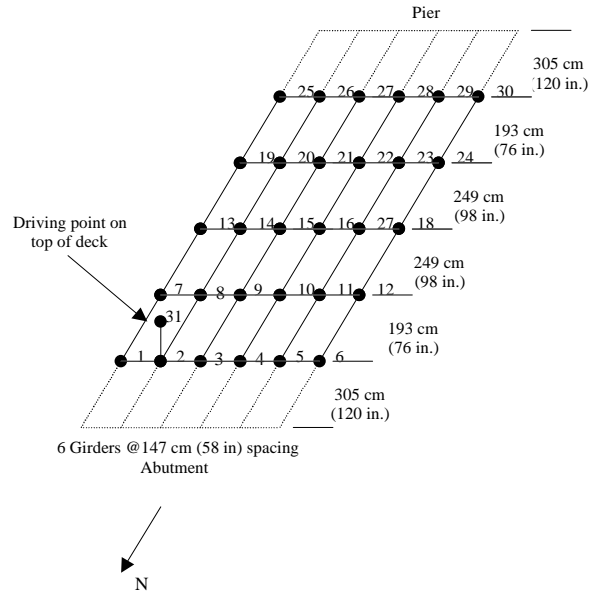


Figure 5. Accelerometer and excitation locations

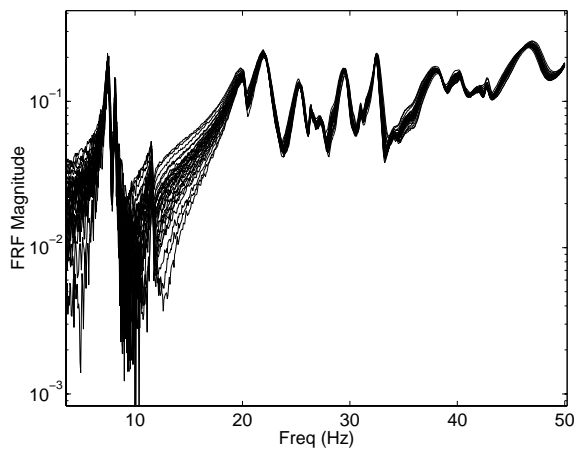


Fig. 6 Thirty FRFs measured at Pt. 2.

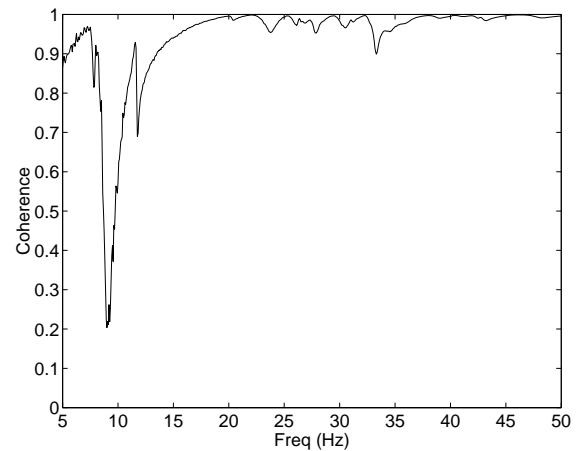


Fig. 8 Coherence function corresponding to the FRF shown in Fig. 7.

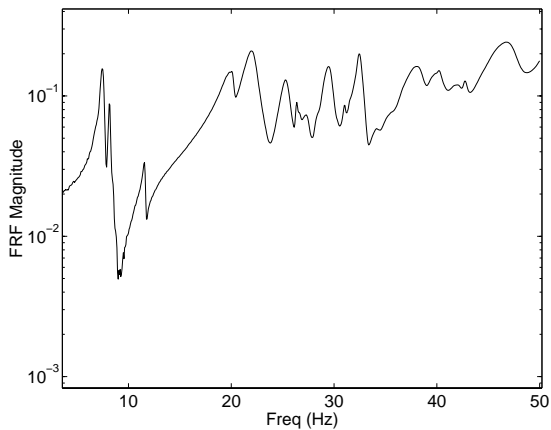


Fig. 7 Average FRF measured at Pt. 2.

The second set of data corresponds to measurements made at two-hour increments over a 24-hr period. Eleven FRFs, each of which are the average of 30 measurements made at the two-hour increments, were analyzed. These 11 FRF were analyzed by the bootstrap procedure. Then the 11 FRFs were averaged along with their corresponding coherence functions to form the data for the Monte Carlo analysis. Therefore, at each DOF, the FRFs and coherence used in the Monte Carlo analysis were based on 330 measurements made over a 24 hr time period. Figure 9 shows an overlay of the 11 FRFs for DOF 2. Figure 10 shows the mean FRF for this same DOF and Fig. 11 shows the corresponding mean coherence function.

From the results presented in [11] it was known that temperature variations during the day produce up to 5% changes in the resonant frequencies of this bridge. These temperature changes represent a systematic error source

inherent in these data. Recall that the Monte Carlo procedure assumes that the error source is random.

Table 2 summarizes the results from the statistical analyses of the two Alamosa Canyon Bridge data sets.

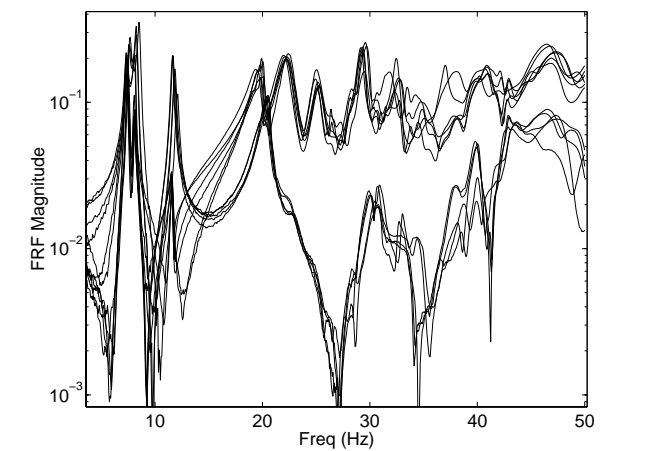


Fig. 9 Overlay of 11 averaged FRFs measured at Pt. 2 over a 24 Hr period.

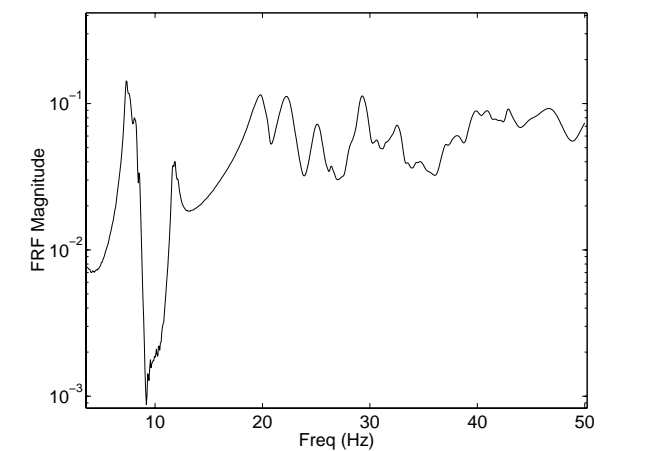


Fig. 10 Average FRF measured at Pt. 2 during a 24 Hr. period.

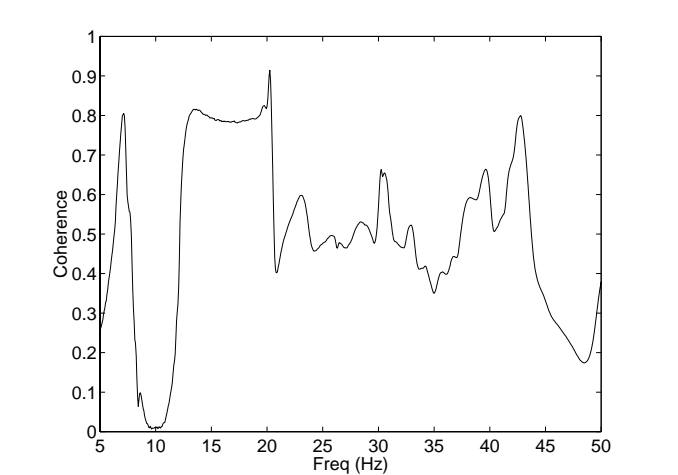


Fig. 11. Coherence function corresponding to the FRF shown in Fig. 10.

### 9. SUMMARY AND CONCLUSIONS

Two methods, Monte Carlo simulation and bootstrap analysis have been presented for evaluating the statistical variations in modal parameters identified from measured FRF data. The variability in these parameters arise from environmental effects such as thermal gradients, service conditions such as traffic loads, and from variability associated with the measurement and data reduction process. Note that some of these processes produce random errors while others produce bias (systematic) errors.

Before modal-based damage identification procedures can be routinely applied to a bridge, particularly in a remote monitoring mode, the effects of these variability sources on the modal-based parameters monitored by the damage identification algorithm must be quantified. Such quantification may require measurements to be made at

TABLE 2 Comparison of Bootstrap and Monte Carlo Statistical Analysis Results on Data from the Alamosa Canyon Bridge				
	Mean (Bootstrap)	95% Conf. Interval (Bootstrap)	Mean (Monte Carlo)	95% Conf. Interval (Monte Carlo)
Individual Test (30 Averages)				
1 <sup>st</sup> Mode Freq.	7.4998 Hz	0.0100 Hz (0.0018 Hz)*	7.4997 Hz	0.0023 Hz
2 <sup>nd</sup> Mode Freq.	8.1341 Hz	0.0064 Hz (0.0012 Hz)	8.1334 Hz	0.0020 Hz
3 <sup>rd</sup> Mode Freq.	11.604 Hz	0.0057Hz (0.0010 Hz)	11.604 Hz	0.0011 Hz
1 <sup>st</sup> Mode Damp.	2.15 %	0.14% (0.026 %)	2.15%	0.024%
2 <sup>st</sup> Mode Damp.	1.08%	0.11% (0.020 %)	1.07%	0.023%
3 <sup>rd</sup> Mode Damp.	0.96%	0.031% (0.0057 %)	0.96%	0.0090%

Data from 11 Tests (30 averages each) Over 24 Hrs.				
1 <sup>st</sup> Mode Freq.	7.3267 Hz	0.101 Hz (0.0305 Hz)	7.3514 Hz	0.0152 Hz
2nd Mode Freq.	8.1016 Hz	0.0673 Hz (0.0202 Hz)	8.1407 Hz	0.0289 Hz
3rd Mode Freq.	11.581 Hz	0.0383 Hz (0.0115 Hz)	11.547 Hz	0.0233 Hz
1 <sup>st</sup> Mode Damp.	2.6%	1.2% (0.36 %)	1.6%	0.26%
2 <sup>nd</sup> Mode Damp.	1.5%	0.42% (0.13 %)	0.64%	0.23%
3rd Mode Damp.	1.3%	0.24% (0.072%)	0.69%	0.16%
* Values in parentheses correspond to bootstrap confidence intervals divided by the square root of the number of bootstrap samples				

different times of the year, during different weather conditions, and when the bridge is experiencing different service conditions. Based on the results of such tests, it is conceivable that bounds can be developed for the modal-based parameters that are monitored by the damage identification system. Damage must cause changes in these parameters that are outside these bounds for a definitive statement to be made regarding the onset of damage in the bridge.

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